Quantum Physics I (8.04) Spring 2016
Assignment 2

Massachusetts Institute of Technology
Physics Department
February 11, 2016

Due Thu. February 18, 2016
5:00pm

Problem Set 2

1. **de Broglie wavelength** [20 points]

   (a) The de Broglie wavelength of a *non-relativistic (nr)* electron with kinetic energy E_{kin} can be written as as

   $$\lambda_{nr} = \frac{\delta}{\sqrt{E_{kin}}}\ \text{Å}\,.$$

   In this formula δ is a unit-free constant, and the value of the energy E_{kin} is entered in eV as a pure number. The answer comes out in Angstroms (Å $= 10^{-10}$m) . Give the value of the unit-free constant δ.

   (b) The de Broglie wavelength of a *relativistic (r)* electron with energy E can be calculated in terms of the γ factor of the electron: $E = \gamma m_e c^2$. One finds

   $$\lambda_r = \frac{\ell}{\sqrt{\gamma^2 - 1}}\ .$$

   What is the value of ℓ in fm $= 10^{-15}$m? Is this a well-known length?

   (c) Rewrite the expression for λ_{nr} in (a) in terms of ℓ and γ, using $E_{kin} = (\gamma - 1)m_e c^2$. Demonstrate that $\lambda_r < \lambda_{nr}$ for any energy.

   (d) A few numerical calculations:

      i. What is the energy of an electron whose de Broglie wavelength is equal to its Compton wavelength? Is that electron relativistic: Is it moving faster than $0.2\,c$?

      ii. The de Broglie wavelength of a particle gives you the rough idea of the distance scale it can explore in a collision experiment. The International Linear Collider, which may be built in the near future, is expected to accelerate electrons to 1 TeV = 1000 GeV. What is the de Broglie wavelength of such electrons? Compare to the de Broglie wavelength of 7 TeV protons at the LHC at Geneva.

      iii. What is the maximum electron *kinetic energy*, and the associated $\beta = v/c$, for which the non-relativistic value of λ (in (a) or (c)) has an error less than or equal to 10% ?

2. **Bohr radius, electron Compton wavelength, and classical electron radius.** [10 points]

   The classical electron radius r_0 is the radius obtained by setting the electrostatic energy associated to a charged ball of radius r_0 equal (up to constant factors) to the rest energy of the electron

   $$\frac{e^2}{r_0} = m_e c^2 \quad \rightarrow \quad r_0 = \frac{e^2}{m_e c^2}\,.$$

   Here e is the charge of the electron. The bar-Compton wavelength $\lambdabar_C$ of the electron is

   $$\lambdabar_C \;=\; \frac{\hbar}{m_e c}\,.$$

   Finally, the fine structure constant α, which measures the strength of the electromagnetic coupling is

   $$\alpha \;=\; \frac{e^2}{\hbar c} \;\simeq\; \frac{1}{137}\,.$$

   (a) The Bohr radius a_0 is the length scale that can be constructed from $e^2, \hbar$, and m_e and no extra numerical constants. Find the formula for the Bohr radius by consideration of units. Evaluate this length in terms of fm.

   (b) Show that the three lengths form a geometric sequence with ratio α:

   $$a_0 \;:\; \lambdabar_C \;:\; r_0 \;=\; 1 \;:\; \alpha \;:\; \alpha^2\,.$$

   Use this to give the values of $\lambdabar_C$ and r_0 in fm.

3. **Two-by-two matrices and linear devices.** [10 points]

   Consider the two-beam Mach-Zender interferometer and a beam represented by the two-component column vector u:

   $$u \;=\; \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\,, \quad \text{with} \quad |u_1|^2 + |u_2|^2 = 1\,.$$

   Any *linear* optical element in the interferometer can be represented by a two-by-two matrix R such that with input u beam the output is a u' beam given by

   $$u' \;=\; R\,u\,.$$

   Show that conservation of probability for *arbitrary* u requires that R be a unitary matrix. A (finite size) matrix R is said to be unitary if $R^\dagger R = \mathbf{1}$, where dagger denotes the operation of transposition and complex conjugation.

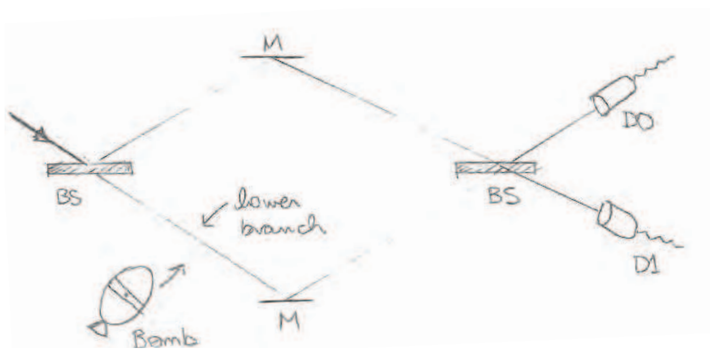
4. **Improving on bomb detection** [15 points]

We modify the Mach-Zehnder interferometer to increase the percentage of Elitzur-Vaidman bombs that can be vouched to work without detonating them. For this purpose we build a beam-splitter with reflectivity R and transmissivity T. A photon incoming (from either port) has a *probability* R to be reflected and a probability T to be transmitted ($R+T=1$). Let r and t denote the *positive* square roots:

$$r \equiv \sqrt{R}, \quad t \equiv \sqrt{T}.$$

(a) Build the two-by-two matrix U that represents the beam splitter. For this consider what happens when a photon hits the beam splitter from the top side (input $\begin{pmatrix}1\\0\end{pmatrix}$) and when it hits it from the bottom side (input $\begin{pmatrix}0\\1\end{pmatrix}$). To fix conventions U will have all entries positive (and real) except from the bottom right-most element (the 2,2 element). Confirm that U is unitary.

The interferometer with detectors D0 and D1 (shown below) uses two identical copies of the beam splitter. The incoming photon arrives from the top side.



(b) A defective bomb is inserted in the lower branch of the interferometer. What are the detection probabilities P_0 and P_1 at D0 and D1 respectively?

A functioning bomb inserted in the lower branch of the interferometer. What is the detonation probability P_{boom} and the detection probabilities P_0 and P_1? Express your answers in terms of R and T.

(c) You test bombs until you are reasonably sure that either they malfunction or that they are operational. What fraction f of the operational bombs can be certified to be good without detonating them? Give your answer in terms of R. What is the maximum possible value for f?

5. **Plane waves for matter particles.** [10 points] Assume we want to represent the wave for a matter particle moving in the x direction with momentum $p = \hbar k$. A reasonable guess for such a wave is

$$\Psi(x,t) = \cos(kx - \omega t) + \gamma \sin(kx - \omega t)\,,$$

where γ is a constant. A physical requirement is that an arbitrary displacement of x or an arbitrary shift of t should not alter the character of the wave. We will demand therefore that after the shift, whose effect is to change the phase by some constant ϵ, we have

$$\cos(kx - \omega t + \epsilon) + \gamma \sin(kx - \omega t + \epsilon) \;=\; a\big[\cos(kx - \omega t) + \gamma \sin(kx - \omega t)\big]$$

for some constant a that may depend on ϵ.

Write the equations that follow from the above requirement. Find the two possible solutions for γ and the associated a. Which is the solution that corresponds to our conventional description of a matter wave?