Quantum Physics I (8.04) Spring 2016
Assignment 3

MIT Physics Department *Due Thu. February 25, 2016*
February 18, 2016 *5:00pm*

Announcements

- Recommended Reading: Griffiths, sections 1.1, 1.2, 1.4 and 1.5.

Problem Set 3

1. **Exercises with commutators** [10 points] Let A, B, and C be linear operators.
   (a) Show that $[A, BC] = [A, B]C + B[A, C]$.
   (b) Show that $[AB, C] = A[B, C] + [A, C]B$.
   (c) Show that $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.
   (d) Calculate $[\hat{x}^n, \hat{p}]$ and $[x, \hat{p}^n]$ for n an arbitrary integer greater than zero.
   (e) Calculate $[\hat{x}\hat{p}, \hat{x}^2]$ and $[\hat{x}\hat{p}, \hat{p}^2]$.

2. **Simple tests of the stationary phase approximation** [10 points]

   In here we consider integrals of the form

$$\Psi(x) \;=\; \int_{-\infty}^{\infty} dk\, \Phi(k) e^{ikx}\,,$$

   where $\Phi(k)$ is a function that is sharply localized around $k = k_0$. In each of the following cases use the stationary phase argument to predict the location of the peak of $|\Psi(x)|$. Then compute the integral exactly to find $\Psi(x)$, $|\Psi(x)|$, and to confirm your prediction.

   (a) $\Phi(k) = e^{-L^2(k-k_0)^2}$, where L is a constant with units of length.
   (b) $\Phi(k) = e^{-L^2(k-k_0)^2} e^{-ikx_0}$, where x_0 and L are constants with units of length.

   Useful integral: Valid for complex constants a and b, with real part of a positive:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} \exp\Big(\frac{b^2}{4a}\Big)\,, \quad \text{when} \;\; \mathrm{Re}(a) > 0\,.$$

3. **Galilean invariance of the free Schrodinger equation.** [15 points]

Show that the free-particle one-dimensional Schrödinger equation for the wavefunction $\Psi(x,t)$:

$$i\hbar\frac{\partial \Psi}{\partial t} \;=\; -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}\,,$$

is invariant under Galilean transformations

$$x' = x - vt\,, \quad t' = t\,.$$

By this we mean that there is a $\Psi'(x',t')$ of the form

$$\Psi'(x',t') \;=\; f(x,t)\,\Psi(x,t)\,,$$

where the function $f(x,t)$ involves $x, t, \hbar, m$ and v, and such that Ψ' satisfies the corresponding Schrödinger equation in primed variables.

$$i\hbar\frac{\partial \Psi'}{\partial t'} \;=\; -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi'}{\partial x'^2}\,.$$

(a) Find the function $f(x,t)$. [Hint: Note that the function $f(x,t)$ cannot depend on any observable of Ψ; it is a universal function that is used to transform *any* Ψ. Thus if Ψ is a (single) plane wave, f cannot depend on its momentum or its energy.]

(b) Demonstrate that the plane wave solution

$$\Psi(x,t) \;=\; A\, e^{i(kx-\omega t)}$$

transforms as expected. In other words, give Ψ' and show that it represents, in the primed reference frame, a particle with the expected momentum and energy.

4. **Re-do current conservation in 3D** [10 points]

In class we derived the expression for the one-dimensional probability current $J(x,t)$ starting from $\rho(x,t) = |\Psi(x,t)|^2$ and using the one-dimensional Schrödinger equation to write

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0\,.$$

Repeat the same steps starting from

$$\rho(\mathbf{x},t) \;=\; \big|\Psi(\mathbf{x},t)\big|^2\,,$$

and using the three-dimensional Schrödinger equation to derive the form of the probability current $\mathbf{J}(\mathbf{x},t)$ that should appear in the conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{J} = 0\,.$$

5. **Time evolution of an overlap between two states.** [10 points] (Merzbacher)

   Consider a wavefunction that at time $t = 0$ is the superposition of two widely separated narrow wave packets Ψ_1 and Ψ_2:

   $$\Psi(x, 0) \;=\; \Psi_1(x, 0) + \Psi_2(x, 0)\,.$$

   Each packet is separately normalizable. We define the overlap integral $\gamma(t)$ as

   $$\gamma(t) \;\equiv\; \int_{-\infty}^{\infty} \Psi_1^*(x, t)\Psi_2(x, t)dx\,.$$

   At time equal to zero the value of $|\gamma(0)|$ is very small. As the packets evolve and spread, what will happen to the value of $|\gamma(t)|$? Will it increase as the packets overlap?

6. **Probability current in one dimension** [10 points]

   Calculate the probability current $J(x)$ for the following wavefunctions, all of which refer to $t = 0$:

   (a) $\Psi(x) = A\,e^{\gamma x}$. Here A is a complex constant and γ is a real constant.

   (b) $\Psi(x) = N(x)e^{iS(x)/\hbar}$. Here $N(x)$ and $S(x)$ are real.

   (c) $\Psi(x) = Ae^{ikx} + Be^{-ikx}$. Here A, B are complex constants and k is real.