

# Quantum Physics I (8.04) Spring 2016

## Assignment 9

MIT Physics Department  
April 21, 2016

Due Friday April 29, 2016  
12:00 noon

### Reading:

Qj cplcp.'Ej cr vgt"33<Uecvgtlpi "cpf "Tguqpcpegu  
Griffiths: section 4.1.

### Problem Set 9

#### 1. A numerical test of stationary phase. [10 points]

We have used stationary phase to figure out the time dependence of the position of peaks in wavepackets constructed from integral representations. More generally, the stationary phase approximation can help get the value of the integral itself.

Consider the integral of a Gaussian peaked at  $x = 2$  against a phase factor:

$$f(\lambda) = \int_{-\infty}^{\infty} dx e^{-100(x-2)^2} e^{i\phi(\lambda,x)}, \quad \phi(\lambda,x) = 50\left(x - \frac{1}{32}\lambda x^4\right), \quad \lambda \in \mathbb{R}.$$

We want to confirm that  $|f(\lambda)|$  peaks at a value  $\lambda_*$  selected by stationary phase and get the value of  $f(\lambda_*)$ .

- What is the width  $\Delta$  at half-maximum for the gaussian? In other words, what is the largest  $\Delta$  for which for all  $x$  in  $|x - 2| \leq \frac{1}{2}\Delta$  the gaussian is larger than half-maximum? If you had to do the integral numerically, would it be safe to integrate from 1 to 3? Explain.
- Use stationary phase to find the critical value  $\lambda_*$  of  $\lambda$  for which  $f(\lambda)$  would have the largest magnitude. For  $\lambda_*$  write  $\phi(\lambda_*, x)$  as a Taylor expansion around  $x = 2$  up to and including terms quadratic in  $(x - 2)$ .
- What is the excursion of the phase  $\phi(\lambda_*, x)$  for  $|x - 2| < \frac{1}{2}\Delta$ ? Your result, expressed in units of  $\pi$ , should imply that it is a decent approximation to ignore the phase *variation* at the critical  $\lambda$ . Do so and then perform the resulting integral analytically. The answer is a complex number. Write your answer in terms of a phase times the magnitude.
- Perform the integral analytically using the quadratic approximation for the phase. Write your answer in terms of a phase times the magnitude.
- Perform the integral numerically as a function of  $\lambda$  for the interval  $\lambda \in [0, 1]$ . Plot the absolute value  $|f(\lambda)|$ . What is the value of  $f(\lambda)$  for the critical  $\lambda$ ? Compare with your previous estimates. What is the value of  $\lambda$  that leads to the largest  $|f(\lambda)|$ ?

2. **Testing Levinson's theorem in an example** [10 points].

For the potential  $V(x) = -V_0$  for  $0 < x < a$ ,  $V(x) = 0$  for  $x > a$  and  $V(x) = \infty$  for  $x < 0$  we calculated in class the phase shift  $\delta(E)$  finding

$$\tan \delta = \frac{1 - \frac{k'}{k} \cot k'a \tan ka}{\tan ka + \frac{k'}{k} \cot k'a},$$

with

$$k^2 = \frac{2mE}{\hbar^2}, \quad k'^2 = \frac{2m(E + V_0)}{\hbar^2}, \quad z_0^2 = \frac{2mV_0a^2}{\hbar^2}$$

- As the energy  $E$  goes to zero,  $ka \rightarrow 0$ . What happens to  $k'a$ ? Show that  $\tan \delta$  goes to zero, and thus we can take  $\delta \rightarrow 0$  as  $ka \rightarrow 0$ .
- What is the limit for  $\tan \delta$  as  $E \rightarrow \infty$ ? Explain in detail.
- Call  $u \equiv ka$  and write  $\tan \delta$  as a function of  $u$  and  $z_0$

$$\tan \delta = f(u; z_0) \quad \rightarrow \quad \delta = \text{ArcTan}[f(u; z_0)].$$

Write the function  $f(u; z_0)$ .

In order to construct plots with Mathematica, I found it difficult to use  $\text{ArcTan}[\dots]$  because it uses the range  $(-\pi/2, \pi/2)$  and the graphs give discontinuities. One option (suggested by W. Taylor) is to differentiate the  $\text{ArcTan}$  function and then to integrate it again! Since  $\delta = 0$  for  $u = 0$  we can write:

$$\delta(u; z_0) = \int_0^u du' \frac{d}{du'} \text{ArcTan}[f(u'; z_0)].$$

Let the computer take the derivative and integrate. If you find a simpler way to do this let us know!

- Graph the phases  $\delta(u, z_0)$  as a function of  $u$  for  $z_0 = 2, 5, 9$ . For  $z = 2$  use  $u \in [0, 15]$ , for  $z_0 = 5$  use  $u \in [0, 20]$  and for  $z_0 = 9$  use  $u \in [0, 30]$ . In each case explain how the result is consistent with Levinson's theorem and state how close is  $\delta$  at the upper value of  $u$  to the expected value of  $\delta(E = \infty)$ .

### 3. Scattering off a step and a wall [10 points]

Consider the potential

$$V(x) = \begin{cases} V_0, & \text{for } 0 < x < a, \quad V_0 > 0, \\ 0, & \text{for } x > a, \\ \infty, & \text{for } x \leq 0. \end{cases}$$

Calculate the phase shift  $\delta(k)$  as a function of  $k$ . You will have to consider two cases:

- $E(k) > V_0$ . Call  $k'$  the wavenumber for  $x < a$ . You may want to do this starting from the beginning (for practice). Otherwise you could try to use the example worked in class (and Problem 2 here), where instead of a step we had a well of depth  $V_0$ , and modify the answer suitably. Leave your answer in the form  $\cot \delta = \dots$
- $E(k) < V_0$ . You may want to do this starting from the beginning (for practice). Otherwise you could try some analytic continuation of the result of part (a). Leave your answer in the form  $\cot \delta = \dots$
- Plot  $\delta(k)$  as a function of  $u = ka \in [0, \infty]$  for a potential with  $z_0 = 5$  (recall that  $z_0^2 = \frac{2mV_0a^2}{\hbar^2}$ ).

### 4. Scattering off a delta function and a wall. [15 points]

Consider our usual one-dimensional potential with  $V(x) = \infty$  for  $x \leq 0$ , and with

$$V(x) = g\delta(x - a), \quad g > 0 \quad x > 0.$$

We scatter particles with mass  $m$  and energy  $E > 0$  off this potential. We have

$$(ka)^2 = \frac{2mEa^2}{\hbar^2}, \quad \lambda \equiv \frac{mag}{\hbar^2}, \quad \text{unit free.}$$

- Calculate the phase shift  $\delta(k)$ . Write the answer in the form

$$\tan \delta = - \frac{\sin^2(ka)}{h(ka; \lambda)},$$

where  $h(ka; \lambda)$  is a function you must determine. Explain how, with  $\delta$  known, one readily finds the amplitude  $A(k)$  multiplying the ‘sin’ function in the wavefunction for  $0 < x < a$ .

- To understand features of  $\tan \delta$  calculate the leading approximation to it for  $ka \ll 1$ . Discuss the  $\lambda$  dependence of the result. For arbitrary  $ka$  what does  $\tan \delta$  become for  $\lambda \rightarrow \infty$ ?
- Plot  $\delta$ , the time delay  $\frac{1}{a} \frac{d\delta}{dk}$ , and  $|A|$  as functions of  $ka \in [0, 10]$  for  $\lambda = 5$ . Do you see resonances? If so, identify the values of  $ka$ , the time delay  $\frac{1}{a} \frac{d\delta}{dk}$  and the magnitude of  $|A|$ . Is the plot of  $\delta$  consistent with Levinson’s theorem?

**5. A few commutators and a few expectation values.** [10 points]

- (a) Calculate the commutators

$$[L_z, x], \quad [L_z, y], \quad \text{and} \quad [L_z, z].$$

- (b) Calculate the commutators

$$[L_z, p_x], \quad [L_z, p_y], \quad \text{and} \quad [L_z, p_z].$$

- (c) Assume  $\psi_0$  is an  $L_z$  eigenfunction. Show that  $p_y$  and  $p_x$  have zero expectation value in the state  $\psi_0$ .
- (d) Assume  $\psi_0$  is an  $L_z$  eigenfunction. Show that  $y$  and  $x$  have zero expectation value in the state  $\psi_0$ .

**6. Angular momentum in spherical coordinates.** [10 points]

- (a) Calculate the nine partial derivatives of the spherical coordinates  $(r, \theta, \phi)$  with respect to the Cartesian coordinates  $(x, y, z)$  expressing your answers in terms of the spherical coordinates.
- (b) Use the above results to write  $L_x, L_y,$  and  $L_z$  as differential operators in spherical coordinates.
- (c) Compute  $L_x^2, L_y^2,$  and  $L_z^2$  as differential operators in spherical coordinates and use your results to derive the expected form of  $\mathbf{L}^2$  as a differential operator in spherical coordinates.

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