PROFESSOR: Here is where the power of this comes when you decide that you're going to invent all possible Hamiltonians at this moment. You've reduced the infinite dimensional space of functions in the line to two points, so you have a two-dimensional vector space, dramatic reduction.

So here we decide, OK, here is the Hamiltonian. And it's going to be a two-by-two matrix, and it better be Hermitian. So what options do I have? Well, Hermitian means transpose complex conjugated gives you back the same matrix. So let's try to parametrize such a matrix.

I could put a0, a real quantity here, and another real quantity in the bottom size. And if they are real, the transpose complex conjugate will remain the same. That's OK. So I could put a0 and a1 here.

I'll do it in a little different way. I'll put a0 plus a3, and a0 minus a3 here. Now, the thing is that a0 and a3 have to be real. So I'll use a0, a1, a2, and a3. And they all should be real.

So here, transpose complex conjugate doesn't affect these things. They are the same. That's good. Here, we can a1 minus ia2. This is a complex number. And the only thing that must happen is that, when I transpose a complex conjugate, I must get the same thing.

So I should put here a1 plus ia2. Because if I transpose this, I will have it on this side. And then I complex conjugate it, and it becomes this term. Similarly, if I transpose this term, it goes here. But then complex conjugated, it becomes that.

So actually, I claim the most general two-by-two Hermitian matrix. Time independent-- you see, all our quantum mechanics this semester has been time independent potentials. So here it's time independent. And now, this is the most general Hamiltonian you could have. That's it. So when you see something like that, you realize that in an hour or two or after some thinking, you will have solved the most general dynamical system with two degrees of freedom in quantum mechanics.

So I will write this as a0 times this matrix, plus a1 times this matrix, plus a2 times this matrix, plus a3 times this matrix. That's exactly what you have in there. Multiply in these constants and add these matrices, and they give you all what we have.

So actually, these are the basic Hermitian two-by-two matrices. And if you multiply them by real numbers, you still are Hermitian. And if you add them, you still are Hermitian. So the most general Hermitian matrix has four parameters.

And it is a space of matrices spanned by these four matrices. They are so famous, these matrices, that this is called sigma 1, this is called sigma 2, and this is called sigma 3. And they're called the Pauli matrices.

Well, but let's put units to these things. We want to write Hamiltonians. So let's make sure we have units that do the job. The Hamiltonian must have units of energy. So we could do a Hamiltonian that has units of energy. So I'll write h omega, which has units of energy, omega 1, sigma 1, plus h-- I'll put it even over 2-- h omega 2, over 2, sigma 2, plus h omega 3, over 2, sigma 3.

Now you would say, well, why didn't you use the first matrix. I could have used the first matrix, but the first matrix is proportional to the identity. We already learned in our course that if you have an extra constant operator in the Hamiltonian, it doesn't change your calculations in any way. You had the Hamiltonian for the harmonic oscillator. It was h omega N plus 1/2. And the 1/2 was an additive constant that never played any important role.

So this would be an additive constant to the energy. It would tell you how you're measuring the energy from what level. So it's not very interesting. You can use it sometimes, but it's definitely not all that interesting.

So I'll do a little variation of this by writing omega 1, h over 2, sigma 1, plus omega 2, h over 2, sigma 2, plus omega 3, h over 2, sigma 3. And then you say, look, that's interesting, OK, I have an omega on this thing. But omega is fine. We know what it is. It's a frequency, 1 over time unit. But this has units of angular momentum.

h bar has units of angular momentum. And the thing that is a little mysterious here is that we seem to have three of them. So maybe somehow this has to do with angular momentum. So let's investigate it a little bit.

Well, they have units of angular momentum. So maybe I can call some first component of angular momentum, h bar over 2 sigma 1, second component of angular momentum, h 1 over 2 sigma 2, and the third component of angular momentum, h bar over 2 sigma 3.

Well, those are just names. But we can try to do a computation with them. We can try to see what is the commutator of Sx with Sy. And happily, these are matrices, so it's a natural thing to do commutators.

So you would have h bar over 2, sigma 1, with h bar over 2, sigma 2, commutator. And it's equal to h bar over 2 times h bar over 2, sigma 1, sigma 2, minus sigma 2, sigma 1.

So it's h bar over 2 times h bar over 2. And let's do this. Sigma 1 is 0, 1, 1, 0. Sigma 2 is 0, minus i, i, 0, minus 0, minus i, i, 0, 0, 1, 1, 0. OK, I have to do all that arithmetic. Happily, this is not that bad. Let's see if I don't make mistakes.

OK, here I get two terms, an i from the first, a 0 here, a 0, and a minus i here-- minus-- and minus i, a 0, a 0, and an i, which is h bar over 2, times h bar over 2, times-- oh, they don't cancel. They seem to cancel, but there's some minus-- it's actually twice-- of those, so 2i minus 2i, 0, 0.

And here we get a 2 cancels this and then i goes out. So I'll have with this factor and i out is i h bar times h bar over 2, times the matrix 1 minus 1, 0, 0. Somehow, it gave that.

h bar over 2, 1 minus 1-- 1 minus 1 is sigma 3. And h bar over 2 sigma 3 is Sz, so this is all Sz. So it's i h bar Sz. So this stuff, Sx, Sy, is giving you i h bar Sz. And that was exactly like angular momentum.

So not only it has the units of angular momentum, it has the commutation relations of angular momentum. Hermitian operators, two-by-two matrices, they used to be r cross p, all these derivatives, complicated stuff. Here it is-- with two-by-two matrices, you've constructed angular momentum.

What we've constructed at this moment is spin 1/2. A whole spin 1/2 system is nothing else than that-- angular momentum and the freedom of having two discrete degrees of freedom. The interpretation that what they have to do is spin up and spin down is something that physicists came up with. But the mathematics was there waiting as the simplest quantum mechanical problem.

Considering [? who wrote ?] the Schrodinger equation, maybe, if he had been more mathematically inclined, he could have discovered, five minutes later, spin. But he wanted to figure out the wave function of the hydrogen atom and scattering and all these very complicated things.

So needless to say, the other commutation relations work out. So if you check that Sy with Sz, you will get i h bar Sx. And if you do finally Sz with Sx, you will get i h bar Sy.

So these two-by-two matrices satisfy this property. And there is a little more to be said. I want to say a few more things about it because it's counter-intuitive and therefore very nice.

Half of the semester in 805 is devoted to spin 1/2. It takes a while to understand it. So I wanted you to see it, at least once. And the problem is the physical interpretation takes time to get accustomed.

So on the other hand, we did write the Hamiltonian. So the Hamiltonian was omega 1 Sx, plus omega 2 Sy, plus omega 3 Sz. And it's there-- S1, S2, S3, second line. And this is the Hamiltonian.

So people write it sometimes as omega dotted with an S vector, as it's saying it has three components, as omega has three components as well. And there's a lot of physics in this Hamiltonian. It's the simplest Hamiltonian, but it actually represents a spin in a magnetic field.

And what it will make it do, this Hamiltonian, if we solve the differential, this two-by-two matrix equation, we will find that the spin starts to precess. That's the origin of nuclear magnetic resonance, spinning, precessing spins, that the machine makes them precess. And they send a signal and you detect the density of different fluids in the body.