PROFESSOR: We spoke about the hydrogen atom. And in the hydrogen atom, we drew the spectrum, so the table, the data of spectrum of a quantum system. So this is a question that I you in general to be aware of. What do we mean by a diagram of the energy levels in a central potential? So this is something we did for the hydrogen atom.

But in general, the diagram of energy eigenstates in a central potential looks like this. You put the energies here. And they could be bound states that have negative energy. They could be even bound states with positive energy, depending on the system you're discussing.

You remember the harmonic oscillator, the potential is naturally defined to be positive. And all these energy states that the harmonic oscillator has represent bound states. They are normalizeable wave functions. In fact, you don't have scattering states because the potential just reaches forever.

So in general, for a central potential, however, the system is shown like that. And we plot here l. But l is not a continuous variable, so we'll put l equals 0 here, l equals 1, l equals 2, l equals 3.

And then you start plotting the energy levels. You solve the radial equation. Remember, the radial equation is a Schrodinger equation, minus h squared over 2m, d second u d r squared, plus V effective of r u equals Eu. And V effective of r is the V of r that your system had, plus a contribution from angular momentum, 2m.

So this radial equation is a collection of radial equations. We've tried to emphasize that many times already. And you solve it first for l equal 0, then for l equal 1, for l equal 2, you go on and on. So you solve it for l equal 0.

And just like as in any one-dimensional problem, for l equals 0, you solve this equation and you find energy levels. So you sketch them like that. Those are the energy levels for l equals 0. This is the ground state of the l equal 0 radial equation.

Now just to remind you, this is the hard part of solving the Schrodinger equation. Because at the end of the day, the psi is u of r over r times some Ylm. So the l that you have here, the term is a Y and the m is arbitrary. In fact, the u doesn't know about the m value.

So here you have these energy levels. And then what happened with this hydrogen atom is you keep solving, of course, for all the l's. And in general, when you solve for l equals 1, you may find some levels like this.

It's a miracle when the levels coincide. There's no reason why they should coincide. They happen to coincide for the hydrogen atom. And that's because of a very special symmetry of the 1 over r potential orbit.

Then for l equal 2, you solve it. And for l equal 3, you solve. And you find these states, and that's the diagram of states of a central potential. For the hydrogen atom, of course, first, all the energies were negative and the energy levels coincide.

So this is the ground state of the l equal 0 radial equation. This is the ground state of the l equal 1 radial equation. This is the ground state of the l equal 2 radial equation. This is the ground state of the whole system. So this is what we call plotting the spectrum in a radial potential problem. And it's a generic form.

So we were doing this for the hydrogen atom last time to try to understand the various orbits. And we had for the hydrogen atom, V of r was a potential like this, minus e squared over r. And then you have sometimes the l contribution that this function diverges towards the origin, 1 over r squared. And by the time you add them together, This is the original potential, so this could be thought as l equals 0 case.

Then if you have some l over here and some other l, maybe like that, these are the various potentials that you get. And in general, you may want to figure out, for example, if you have an energy level, some particular energy, what are the turning points. So let's consider for that case that's just one curve that we care about.

And the electron will be going from some value of the radius, so this is the plot of the effective potential of the function of radius. And we'll go from one to another. They could be called r minus to r plus.

And our semi-classical interpretation, which is roughly good if you're talking about high quantum numbers, high principle quantum numbers, high l quantum numbers, is that you have an ellipse and the radial distance to the center where the proton is located goes from r plus to r minus. The electron is bouncing back and forth. That is the classical picture.

In the quantum mechanical picture, you expect something somewhat similar. There's going to be a wave function, maybe a wave function here, psi squared. And it's going to be vanishingly small before this point. And then by the time you get here, it's going to be very fast, and then decay again. So the probability distribution will mimic the time spent by the particles, as we

used to argue before.

So let's do a little exercise of calculating the turning points. So how do we do that? Well, we set h squared l times l plus 1 over 2mr squared, minus e squared over r-- that's the effective potential-- equal to the energy of some level n, principle quantum number n. So it would be minus e squared over 2a0, 1 over n squared. That's the value of the energy En.

And the solutions of this quadratic equation are going to give us the r plus and the r minus of the orbit. So it's probably worthwhile to do a little transformation and to say r equal a0 times x, where x is unit free. And then the equation becomes h squared, l times l plus 1, over 2ma0 squared, times l times l plus 1, over x squared, minus e squared over a0x, is equal to minus e squared over 2a0, 1 over n squared.

So the unit should work out. We should get a nice equation without units. So what must be happening is that the coefficient in front of here, h squared 2ma0 squared, let's take the other a0 and separate it out, and transform this. Remember, a0 was h squared over m e squared.

So here we get h squared over 2ma0. And that a0 now has an h squared. And there's m e squared. So the h squared cancels, the m cancels. And this is e squared over 2a0. So this whole coefficient is e squared over 2a0, which is nice because now the e squared over a0, the e squared over a0, a squared over a0, cancel.

And we get l times l plus 1 over x squared-- I've canceled all this factor-- minus 2 over x is equal to minus 1 over n squared. So I multiplied by the inverse of this quantity. It clears up the factor in the first term. It produces an extra factor of 2 in the second term. And you've got a nice simple quadratic equation.

AUDIENCE: Question?

PROFESSOR: Yes?

AUDIENCE: Where do you get the extra [INAUDIBLE] l, l plus 1 [? come from? ?]

PROFESSOR: Oh, there's no such thing. There's too many of them. Thank you very much. Yes, too many, 1 over x squared. Thank you. So let's move this to the other side-- plus 1 over n squared equals 0.

So this is the main equation. And we can write this-- well, the solution for 1 over x is a

quadratic equation in 1 over x. So I'll write it here. 1 plus/minus square root of 1 minus l times l plus 1, over n squared, divided by l times l plus 1. That's just from the quadratic formula.

So then you invert it. So x is now l times l plus 1, over 1 plus/minus square root of 1 minus l, l plus 1 over n squared. And we multiply by the opposite factor to clear the square root. So 1 minus/plus square root of 1 minus l times l plus 1 over n squared.

And the same factor here-- 1 minus/plus square root of 1 minus l, l plus 1 over n squared, all of these things. But that's not so bad. You get l times l plus 1, times this factor that you had in the numerator-- that still is the same-- 1 minus l, l plus 1 over n squared.

Now we're after an interesting piece of information, the two sizes of the ellipse, so it's worth simplifying what you got. This equation is not nice enough, so we're simplifying. And in the denominator, you have a plus b times a minus b. So it's 1 minus 1 minus l, l plus 1, over n squared.

And the good thing is that the denominator, the 1 cancels. And you get the plus, l times l plus 1 over n squared, that cancels, that one. So that, at the end of the day, we get a pretty nice formula.

The formula is x is equal to n squared, 1 minus/plus, 1 minus l times l plus 1, over n squared, like that. So if we wish, it's r plus/minus is x multiplied by a0, so it's n squared a0, 1 minus/plus square root of 1 minus l times l plus 1, over n squared.

OK, so the ellipse is defined by those two values that we have here. But the surprising thing is that the sum over the ellipse, and its eccentricity, is dramatically affected by the values of l. In fact, for l equals largest, l will be comparable to n. Remember, l can go up to n minus 1.

So at that point, this is essentially 1. You cancel this and there's nothing here. So r plus and r minus become about the same for l equals n minus 1. r plus is almost the same as r minus And the orbit is circular, completely circular.

On the other hand, for l equals 0, the orbit is completely elliptical in that the radius for l equal 0, this is 0, 1, plus/minus 1. So sometimes it's twice this value, sometimes it's 0. So you have the case where r minus can be 0, and you have just an orbit that is like that, r minus 0, so extremely elliptical-- elliptical. Of course, that is the semi-classical approximation. So it's more reliable when you have a reasonable l.

And finally we can say here, for example, one interesting thing, that r plus plus r minus, over 2, which is this-- r plus plus r minus is the total longest axis of the ellipse, divided by 2, the center of the ellipse, not the focus, that distance is independent of l, so you have n squared a0.

So a typical Rydberg atom will have n equal 100. So this is an example-- n equal 100, l equals 60, in which case, for n equal 100, r plus/minus is equal to 10,000a0, n squared a0, times this factor, which in one case is 1.8 and in the other case is 0.2. That's what you get for l equal 60 and n equal 100. So this orbit, you have r plus about 18,000a0, and r minus about 2,000a0.

And all of our orbits satisfy this property that if you have this, r minus and r plus, all of the orbits with different l have the same r plus plus r minus. So if this is the total length of the major axis when the orbit becomes circular, it's the same. And this distance, when the orbit becomes very elliptical, is the same as well.

I mentioned last time that this nice property is degeneracy. We're here, if you're keeping n fixed but changing l, you're going from all these ellipses. This is for l equals n minus 1. And this one is for l equals 0. And all this ellipses arc here.

And those are all the semi-classical picture of those degenerate states in the diagram of the hydrogen atom. The diagram of the hydrogen atom was something like this. And you're looking at all the degenerate states that you have there. And they are degenerate, and you would say, well, why are ellipses that look like that degenerate.

Well, even Kepler apparently knew that in Kepler's laws. That he observed that the period of motion of an orbit just depended on the semi-major axis. So periods are related to energies. And it's reasonable that we have this thing in quantum mechanics.

Now this degeneracy, I want to just finish up by emphasizing what you have here. When somebody asks what is the number of states you have here, well, you have to be precise in what you're counting. The number of full physical states of the quantum system is one here, one here, one here. But each one of this corresponds to l equals 1, so each one of this is triply degenerate, because m can be minus 1, 0, and 1.

So here-- three states, three states, three states. Here-- this is five states, five states, five states, because they all have l equal 2. And l equal 2 goes m from minus 2 to plus 2. So we don't actually put three things here. I think that would be confusing. We could not put five and we cannot see. But it should be remembered that there's the implicit extra degeneracy here

associated with the azimuthal quantum number that we sometimes just don't represent it in a figure.