PROFESSOR: How do we state the issue of normalization? See, the spherical harmonics are functions of theta and phi. So it makes sense that you would integrate over theta and phi-- solid angle. The solid angle is the natural integration. And it's a helpful integration, because if you have solid angle integrals and then radial integrals, you will have integrated over all volume.

So for this spherical harmonic, solid angle is the right variable. And you may remember, if you have solid angle, you have to integrate over theta and phi. Solid angle, you think of it as a radius of one. Here is sine theta.

So what is solid angle? It's really the area on a sphere of radius one. The definition of solid angle is area over radius squared, the part of the solid angle that you have.

If you're working with a sphere of radius one, it's the area element is the solid angle. So the area element in here would be, or the integral over solid angle-- this is solid angle, d omega-- you would integrate from theta equals 0 to theta equals pi of sine theta, d theta. And then you would integrate from 0 to 2 pi of d phi.

Now, you've seen that this equation that began as a differential equation for functions of theta ending up being the differential equation for a function of cosine theta. Cosine theta was the right variable. Well, here it is as well, and you should always recognize that. This is minus d of cosine theta.

And here you would be degrading from cosine theta equals 1 to minus 1. But the minus and the order of integration can be reversed, so you have the integral from minus 1 to 1 of d cos theta and then the integral from 0 to 2 pi of alpha. So this is the solid angle integral. You integrate d of cosine theta from minus 1 to 1 and 0 to 2 pi of d phi.

And we will many times use this notation, the omega, to represent that integral so that we don't have to write it. But when we have to write it, we technically prefer to write it this way, so that the integrals should be doable in terms of cosine theta. So what does this all mean for our spherical harmonics? Well, our spherical harmonics turned out to be eigenfunctions or Hermitian operators. And if they have different l's and m's, they are having different eigenvalues.

So eigenfunctions of Hermitian operators with different eigenvalues have to be orthogonal. So we'll write the main property, which is the integral of 2l, say l prime, m prime, of theta and phi. And he put the star here. I'll put the star just at the Y, and he's complex conjugated the whole thing.

Remember that in our problem, one wave function was complex conjugated. The other wave function is not complex conjugated. They're different ones because l and m and l prime and m prime could be different. So orthogonality is warranted. Two different ones with different l's and different m's should give you different values of this integral.

So at this moment, you should get a delta l prime l delta m prime m. And if l and m are the same as l prime and m prime, you have the same spherical harmonic. And all this tremendous formula over there, with 2l plus 1 and all these figures, you're guaranteed that in that case you get 1 here. So this formula is correct as written. That is the orthonormality of this solution.

Probably, this stage might be a little vague for you. We saw this a long time ago. We may want to review why eigenfunctions of Hermitian operators with different eigenvalues are orthogonal and see if you could prove. And you do it, or is it kind of a little fuzzy already? We saw it over a month ago.

So time to go back to the Schrodinger equation. So for that, we remember what we have. We have minus h squared over 2m Laplacian of psi plus V of r psi equals E psi. And the Laplacian has this form, so that we can write it the following way-- minus h squared over 2m 1 over r d second dr squared r psi-- I won't close the brackets here-- minus this term.

So I'll write it minus one over h squared r squared l squared psi plus V of r psi is equal to E psi. Now, you could be a little concerned doing operators and say, well, am I sure this l squared is to the right of the r squared? r and l-- l has momentum. Momentum [INAUDIBLE] with r. Maybe there's a problem there.

But rest assured, there is no problem whatsoever. You realize that l squared was all these things with dd phetas and dd phis. There was no r in there. It commutes with it. There is no ambiguity.

We can prove directly that l squared commutes with r, and it takes a little more work. But you've seen what l squared is. It's the dd thetas and dd phis. It just doesn't have anything to do with it. So now for the great simplification.

You don't want any of your variables in this equation. You want to elicit to a radial equation. So we try a factorized solution. Psi is going to be-- of all the correlates-- is going to be a product a purely radial wave function of some energy E times a Ylm of theta and phi. And we can declare success if we can get from this differential equation now a radial differential equation, just for r.

Forget thetas and phi. All that must have been taken care of by the angular momentum operators. And we have hoped for that. In fact, if you look at it, you realize that we've succeeded. Why?

The right-hand side will have a factor of y and m untouched. V of r times psi will have a factor of Ylm untouched. This term, having just r derivatives will have some things acting on this capital R and Ylm untouched. The only problem is this one. But l squared on Ylm is a number times Ylm. It's one of our eigenstates.

Therefore, the Y's and m's drop out completely from this equation. And what do we get? Well, you get minus h squared over 2m 1 over r, d second, dr squared, r capital RE minus l squared on psi lm-- or Ylm now-- is h squared times l times l plus 1. So the h squared cancels. You get l times l plus 1 r squared, and then we get the RE of r times the psi lm that has already-- I started to cancel it from the whole equation.

So I use here that L squared from the top blackboard over there, has that eigenvalue, and the psi lm has dropped out. Then I have the V of r RE equals E time RE of r. So this is great. We have a simplified equation, all the angular dependencies gone.

I now have to solve this equation for the radial wave function and then multiply it by a spherical harmonic. And I got a solution that represents a state of the system with angular momentum l and with z component of angular momentum m. The only thing I have to do, however, is to clean up this equation a little bit. And the way to clean it up is to admit that, probably, it's better as an equation for this product.

So let's clean it up by multiplying everything by r. dr squared of little r RE. If I multiplied by r here, I will have plus h squared over 2m l times l plus 1 over r squared rRE plus V of r times r times RE equals E times r times RE of r. So we'll call u of r r times RE of r, and look what we've got. We've got something that has been adjusted, but things worked out to look just right. Minus h squared over 2m d second, dr squared u of r plus-- let me open a parentheses V of r plus h squared over 2m r squared, l times l plus one u of r is equal to E times u of r.

Here it is. It's just a nice form of a one-dimensional Schrodinger equation. The radial equation for the wave function dependents, a long r, has become a radial one-dimensional particle in that potential, in which you should remember two things. That this u is not quite the full radial dependent. The radial dependent is RE, which is u over r.

But this equation is just very nice. And what you see is another important thing. If you look at the given particle in a potential, you have many options. You can look first for the states that have 0 angular momentum-- l equals zero-- and you must solve this equation.

Then you must look at l equals 1. There can be states with l equals one. And then you must solve it again. And then you must solve for l equals 2 and for l equals 3 and for all values of l. So actually, yes, the three-dimensional problem is more complicated than the one-dimensional problem, but only because, in fact, solving a problem means learning how to solve it for all values of l.

Now, you will imagine that if you learn how to solve for one value of l, solving for another is not that different. And that's roughly true, but there's still differences. l equals 0 is the easiest thing.

So if the particle is in three dimensions but has no angular momentum-- and remember, l equals 0 means no angular momentum-- it's this case. l equals 0 means m equals 0. l squared is 0. lz is 0. This is 0 angular momentum.