PROFESSOR: So we'll look at the Schrodinger equation. We're going to work for bound states. We could work with scattering states in spherical coordinates, but it's usually done in advanced courses.

The most important thing is the calculation of the bound states. This is kind of a neat part of quantum mechanics because, in a sense, you get accustomed in quantum mechanics about uncertainties. You cannot predict the probability that the photon will go through this branch of the interferometer or this other probability. You can't be sure and all this. But here you get the energy levels. And you get the exact energy level. So it's a nice thing that, in quantum mechanics, energy levels are things that you calculate exactly.

Of course, when you try to measure energy levels experimentally, the uncertainties arise again. You get the photon of some energy and then there's some energy time uncertainty or something like that, that can bother you. But the end result is that these systems have beautifully the term in fixed numbers called energy levels. And that's what we're aiming at.

It's a nice thing because it's the most physical example. It has many applications. The energy levels of hydrogen atom. The more you study, the more complicated they are because you can include fine effects, like the effects of the spins of the particles. What does it do? The specs of relativity. What does it do? All kinds of things you can put in and do more and more accurate results. So it's really unbelievable how much you can learn with the fine spectrum of the hydrogen atom.

Our equation is minus h squared over 2m, d second dr squared-- the radial equation-- plus h squared l times l plus 1 over 2mr squared minus ze squared over r, u is equal to Eu. So this is the radial equation. Remember it was like the Schrodinger equation for a variable u. And the wave function was u over r times psi lm or Ylm, actually, a spherical harmonic.

Here I could have labeled u with E and l because could certainly u depends-- u is u of r. And u depends on the energy that we're going to get. And it will depend on l that is there, in the differential equation.

This is the effective potential that we discussed before. It was the original potential to which you add the centrifugal barrier. And what are you supposed to solve here? You're supposed to solve for l equals 0. To find some states for a, go one, two, three, four, infinity. You should find all the energy levels of this thing.

So the wave function is psi of r theta and phi will be this u of r over r times Ylm of theta and phi. And, in fact, plugging that into the Schrodinger equation was what gave us this radial equation. So this was called the radial equation, which we talked about, but never quite solved it for any particular example.

OK. As you know, we like, in this course, to get rid of units and constants, so our first step is to replace r by a unit free x. And we have the right quantity. a0. a0 would be the perfect thing. So you-- if you do that, you will be able to clean up the units. But if you do that, you might not quite be able to clean up the z from all the places that you would like to clean it up. So we can improve that-- maybe it's not too obvious. --by putting a 2 over z Here. z has no units, of course, so that you would probably do by trial and error. You would say, well I want to get rid of the z as well. And I have it in this form. And you will see how it works out.

So at this moment, I have to plug r into this equation and just clean it up. And what should happen is that everything, all the units, should give you a factor with units of energy. Because at the end of the-- whatever is left, it's not going to have units. Everything out must have units of energy.

So this takes about one line to do. I'll skip some algebra on these things. I'll try to post notes soon on this. So if you leave a line you could do that calculation for yourselves and it might be worth it.

The claim is that we get 2z squared e squared over a0, multiplying minus d second, dx squared plus l times l plus 1 over x squared minus 1 over x, times u, equals Eu. That will be the common factor that will come out of everything by the time you solve it. And it has the units of energy, as you can imagine. e squared over a0 has a unit of energy.

So might as well move that to the other side to get the final form of the equation, which would be minus d second, dx squared plus l times l plus 1 over x squared, minus 1 over x, u is equal to minus kappa squared u. Where kappa squared is going to be minus E over 2z squared little e squared over a0.

OK. The trick on all these things is to not lose track of our variables. And that-- it's a little challenging. So let me just re-emphasize what's going on here.

We've passed from r variables to u variables, that we will still do more things. And we have a0 in there, z in there, and then the energy is really encapsulated by kappa squared here. And this kappa is unit free.

So if you know kappa, you know the energies. That's what you should be looking for. Kappa is the thing you want to figure out. Knowing kappa is the same as knowing their origins because this is just a constant that gives you the scale of the energy. So this is what we want to solve.

And the equation doesn't look all that complicated, but it's actually not yet that simple, unfortunately. So we have to keep working with it a bit.

So one reason you can see that an equation like that would not be too simple, is to look at it and see what kind of recursion relation it would give you. And that's a good thing to look at the beginning. And you say, OK here I'm going to lower the number of powers by two. Here I'm going to lower the number of powers by one. And here I'm not going to lower the number of powers. You're gonna have three terms. So it's not that simple recursion relation, which the next term is determined by the previous one. So that suggests you better do some work still with this equation to simplify the situation.

And several things that you can do-- one thing is to look at the behavior of the equation, near infinity, near zero, and see if you see patterns going on.

One thing we're going to do is to-- which is not urgent, but it's usually done. And people do it in different ways. --is to look at x goes to infinity and see what the solutions may look like. So as x goes to infinity, the differential equation probably can be approximated to keep this storm to see how things vary. And x goes to infinity. Throw this, throw this, and keep that. So you would have d second u dx squared is equal to kappa squared. So this suggests that u goes like e to the plus minus kappa x.

So exponential behavior. e to the plus minus kappa x. Ideally, of course, for our solutions, we would like the minus one, but we will see what the equations do. Now this suggests, yet another transformation that people do, which is-- look, kappa is dimensionless and we had x that is unit free also. So kappa has no units, x has no units. So let's move to yet another variable.

We started with r being proportionate to x. And now we can put factors that may help us without units here. So I'm not suggesting that this is something that would occur to me, if I'm doing this problem, but it's certainly a possible thing to do. To say, OK, I'm going to define now rho as kappa x. And with x being given by this, rho would be 2 kappa z over a0 r.

So rho is going to be my new coordinate. I'm sorry, we've gone to x. And now we've gone to rho, a new variable over here.

So what happens to the differential equation? Well, it's going to be a little better, but in particular the solutions may be a little better. But here it is. If you look at the differential equation. The differential equation here. Think of moving the kappa squared here below. And then you see, immediately, it fits perfectly well in the first two terms. So you get minus d second, d rho squared plus l times l plus 1, rho squared. And here it doesn't fit all that well. You would have 1 kappa leftover, so 1 over kappa rho, u equals minus u.

Yes it's kind of suggestive. The kappa has almost disappeared from everywhere, but it better not disappear from everywhere. If it would have disappeared from everywhere we would have been in problems because the equation would've not fixed the energy. You see we're hoping that the differential equation will have additional solutions for some values of the energy and for the others no. So the energy better not disappear. It came close to disappearing, the kappa, but it's still here, so we're still OK.

And now, you will say, OK, this is nice. If you look at rho going to infinity again. It works as we wanted. You get d second u, d rho squared is equal to u. And that means u goes to e to the plus minus rho, which is what inspired this.

And the other part of the solution is the solution at near r going to 0. Or your x going to 0 or near rho equal-- going to 0. So near rho going to 0, you have these two terms. And we actually did it last time. We analyzed what was going on with this equation last time. Near rho equal to 0. And we found out that u must behave like rho to the l plus 1 for rho going to 0.

So it was from this two terms the differential equation. And for rho going to 0. Remember the wave function must vanish. And how fast it should vanish? Should vanish to this power to the l plus 1.

OK. So you know lots of things about this function. So first of all, that this thing doesn't have necessarily polynomial solutions because it behaves exponentially. Moreover, you know it doesn't start with constant plus rho plus rho squared. It starts with rho to the l plus 1. So it is pretty important to see all these things before you try to do a recursion relation because recursion relation might lead you to funny things.

So here we go. What do we do based on all this? We try something better. Which is-- we said u of rho, the solution-- we're writing ansets --is going to be rho to the l plus 1, that will have the right behavior for rho going to 0. An unknown function w of rho times e to the minus rho, which is the right behavior we want.

Now there is no assumption, whatsoever, when you write in ansets of this form. You're just expressing your knowledge. Because at the end of the day if rho-- if omega-- or w here, actually, is undetermined. This w will have an e to the rho that cancels this factor. And may be off with some funny power that cancels this. This is just a hope that we're expressing that by writing this solution in this form, this quantity may be simple. Because we know this is present in the solution for larger rho, this is present for small rho, well in between we might have that.

And when you write analysis of this form, you hope for a simple differential equation for w. So what do we get? We can plug that ansets into the differential equation that we already have and see what happens. And indeed that's what we'll do.

I'll skip the calculation. In my notes, it took me half a page and I write big, so it's not too long.

So what do we get? You get an equation that doesn't look that simple. I'm sorry. It just looks like a step back, but it's not. d second w, d rho squared plus 2 times l plus 1 minus rho. Strange. dw, d rho plus, 1 over kappa minus 2 times l plus 1, w equals 0.

OK. Aesthetically, it looks worse. Certainly that equation on the left board looked nicer, but actually it's pretty good because, again, now look at your recursion relation. How will it be? If you take some power-- fixed power. Here you lose one power. Here with the l plus 1 of these, you lose one power. Here you lose nothing. And here you lose nothing. So you have either one power less or your power. So it's a one step recursion relation without the gap. It's not like the two steps that we had for the harmonic oscillator, for the Legendre polynomials. Here is one step recursion relation. ak plus 1 determined by ak. So we say, excuse me to the equation, you don't look that good, but you're very solvable. So we can proceed.